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COHOMOLOGY OF LOCAL SYSTEMS COMING FROM p-ADIC HYPERPLANE ARRANGEMENTS

BY

GIL ALON

Department of Mathematics and Statistics, McGill University
805 Sherbrooke St. W., Montreal, Quebec H3A 2K6, Canada
e-mail: alon@math.mcgill.ca

ABSTRACT

For a p-adic hyperplane arrangement in a vector space V, we consider a local system of De Shalit on the Bruhat-Tits building of $PGL(V)$. We express this local system in terms of Orlik-Solomon algebras, and calculate its cohomology in the case where the arrangement is finite.

1. Introduction

Let V be a finite dimensional vector space over a field K, and let S be an arrangement of hyperplanes in $V^* = Hom_K(V, K)$. In other words, S is a finite subset of $\mathbb{P}(V)$. When $K = \mathbb{C}$ (or any field of characteristic 0), following the work of Arnold, Brieskorn, Orlik and Solomon (see [O-S]), the deRham cohomology of the complement

$$M(S) := V^* \setminus \bigcup_{\eta \in S} \eta$$

has a simple combinatorial description as the Orlik–Solomon algebra of S, denoted $A_{OS}(V, S)$, which is defined by generators and relations.

New phenomena occur when K is a p-adic field. Assume that $K = \mathbb{Q}_p$, and let L be a $\mathbb{Z}_p$-lattice in V. Consider the quotient map

$$\varphi_L \colon \mathbb{P}(V) \to \mathbb{P}(L/pL).$$

We obtain a hyperplane arrangement $\varphi_L(S)$ in $(L/pL)^*$, which is a vector space over a finite field. Allowing the lattice L to vary, we obtain a collection of "local viewpoints" of the arrangement.

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Let $\mathcal{T}$ be the Bruhat–Tits building of $PGL(V)$. It is a simplicial complex, whose vertices consist of the homothety classes of the lattices L. E. de Shalit in [dS] attached to such an arrangement, and to any simplex σ of $\mathcal{T}$, a certain graded anti-commutative algebra $A(S,\sigma)$, whose defintion is inspired by that of the Orlik–Solomon algebra of a hyperplane arrangement. For a vertex of $\mathcal{T}$, $v_0 = [L]$, we actually have $A(S, v_0) = A_{OS}(L/\pi L, \varphi_L(S))$, and for a higher dimensional simplex σ, $A(S,\sigma)$ is an algebra which interpolates the algebras at the vertices of σ. There are also natural maps $A(S,\sigma) \to A(S,\tau)$ whenever σ is a face of τ, that make the collection $\{A(S,\sigma)\}_{\sigma\in\mathcal{T}}$ into a *local system* $\underline{A}(S)$.

De Shalit conjectured that the cohomology of this local system vanishes in all the positive degrees, and proved it when $dim(V) \leq 3$ or the hyperplane arrangement S is the whole of $\mathbb{P}(V^*)$. In the latter case, the existence of a $GL(V)$-action on the arrangement is very helpful, and one uses tools from the theory of smooth representations of reductive groups. This result is the main technical ingredient in his calculation of the cohomology of Drinfeld's p-adic symmetric space.

The purpose of this work is to prove the following theorem:

THEOREM 1.1: *Let S be a finite hyperplane arrangement in V. Then we have*

$$H^i(\mathcal{T}, \underline{A}(S)) = \begin{cases} 0, & i > 0, \\ A_{OS}(V,S), & i = 0. \end{cases}$$

We expect applications of this theorem to come in the study of p-adic hyperplane arrangements. For example, such arrangements may be obtained from arrangements over a number field (and, in particular, Coxeter arrangements) by passing to the various completions of the field. Given a p-adic hyperplane arrangement, the above vanishing theorem allows to draw conclusions about the arrangement from already known theorems about its images over a finite field. This theorem may also supply interesting resolutions of the Orlik–Solomon algebra as a module over the free exterior algebra on the symbols $\{e_a : a \in S\}$, or as a G-module for a finite linear group G acting on the arrangement. We hope to address these issues in future work.

We start this article by giving an equivalent definition of the local system $\underline{A}(S)$, using tensor products of Orlik–Solomon algebras, thus making the connection between the two constructions a concrete one.

Second, we review the notion of p-adic norms on V, whose dilation classes comprise the Euclidean realization of the Bruhat–Tits building. We define a notion of independence of vectors modulo a norm, and show that the set of norms modulo which a given set of vectors is independent can be contracted to a point. In doing so we are led to introduce a new way of moving around the building. The more standard way of moving along geodesics is not applicable on the subsets of the building we consider, because of their non-convexity.

Then we proceed to the proof of the main theorem cited above. The proof is done by induction on the size of S. In order for our induction process to work, we are led to introduce a larger class of local systems, which depend on the set S and on some additional information. The inductive step is done using a restriction–deletion exact sequence for our local systems, which we transpose from the classical setting of Orlik–Solomon algebras to our setting.

Finally, we treat the case $|S| = 0$, which is the base of our induction process, for our general class of local systems. In that case the local systems become constant on their support. The support of a local system in our class may not be the whole building but rather a nontrivial subcomplex. However, we prove that the support is always contractible, and this allows us to verify the cohomology calculation.

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Added in proof: In a recent preprint, E. Grosse-Kloenne proves de Shalit's acyclicity conjecture for infinite arrangements as well. See [GK]. His methods are different from ours, and we find our geometric approach of independent interest.

2. The Bruhat–Tits building and its euclidean realization

2.1. DEFINITIONS. Throughout the paper, K will be a finite extension of $\mathbb{Q}_p$, endowed with a non-archimedean norm $|\cdot|$, and π a uniformizer. We let V be a $(d+1)$ dimensional space over K, and $G = PGL(V)$.

Definition 2.1: A **lattice in** V is an O_K-submodule of V of rank $(d+1)$. Two lattices L_1 and L_2 are called **equivalent** if $L_2 = uL_1$ for some $u \in K^*$ (in that case we can take u to be an integer power of π).

Definition 2.2: $\mathcal{T}$, the Bruhat–Tits building **of** G is a simplicial complex in which the vertices are the equivalence classes $[L]$ of lattices in V, and the k-simplices are the sets $\{[L_0], [L_1], \ldots, [L_k]\}$ where $L_0 \supset L_1 \supset \cdots \supset L_k \supset \pi L_0$ (strict inclusions).

Remark 2.1: The above inclusions determine a canonical ordering of the vertices in any simplex, up to a cyclic change. A choice of distinguished vertex inside the simplex (also called the leading vertex) determines completely an ordering of the vertices, and, in particular, an orientation of the simplex. We denote by $\hat{\mathcal{T}}_k$ the set of ordered k-simplices. Thus:

$$\hat{\mathcal{T}}_k = \{([L_0], \dots, [L_k]) : L_0 \supset L_1 \supset \cdots \supset L_k \supset \pi L_0\}.$$

Remark 2.2: For a vertex $v = [L_0]$ of $\mathcal{T}$, the neighbouring vertices of v are in one-to-one correspondence with the nontrivial subspaces of $L_0/\pi L_0$, where the latter is viewed as a vector space over the finite residue field $O_K/\pi O_K$. Hence the k-simplices in which v is a vertex are in one-to-one correspondence with flags of subspaces of length k in this finite dimensional vector space. In particular, we see that $\mathcal{T}$ is locally finite.

The Euclidean realization. Since $\mathcal{T}$ is a simplicial complex, we may consider its Euclidean realization, $|\mathcal{T}|$, obtained by gluing realizations of its simplices. Goldman and Iwahori ([G-I]) found a concrete description of this space in terms of *norms*.

Definition 2.3: A norm ρ on V is a function $\rho\colon V \to \mathbb{R}^{\geq 0}$ satisfying:

(1) $\rho(x+y) \leq sup(\rho(x), \rho(y))$,

(2) $\rho(ax) = |a|\rho(x)$ for $a \in K, x \in V$,

(3) $\rho(x) = 0 \Leftrightarrow x = 0$.

Definition 2.4: Two norms ρ_1 and ρ_2 on V are called equivalent if there is a real positive constant c such that $\forall v \in V, \rho_1(v) = c\rho_2(v)$. We denote the equivalence class of a norm ρ by $[\rho]$.

The topological realization $|\mathcal{T}|$ can be taken to consist of all the equivalence classes $[\rho]$. Let us review this correspondence (cf. [G-I], prop. 1.5 on page 142 and prop. 1.7 on page 144).

A flag of lattices $L_0 \supset \cdots \supset L_{k-1} \supset L_k = \pi L_0$ representing a simplex $\sigma = \{[L_0], \dots, [L_{k-1}]\}$ and a sequence of positive numbers $x_0 > x_1 > \cdots > x_{k-1} > x_k = |\pi| x_0$ give rise to a norm ρ defined by: $\rho(v) = |\pi|^i x_j$ where i and j are determined uniquely by $\pi^{-i} v \in L_j \setminus L_{j+1}$. The set of all dilation classes of norms obtained from σ in this way obviously forms an open $(k-1)$-dimensional simplex, and will be denoted by $|\sigma|^o$.

On the other hand, any norm ρ determines an increasing continuum of lattices $L(r) = \{v \in V : \rho(v) \leq r\}$. Since $L(|\pi| r) = \pi L(r)$, there is only a finite number of jumps from $L(1)$ (say) to $L(|\pi|)$, and one obtains a unique lattice flag σ such that $\rho \in |\sigma|^o$.

Definition 2.5: For a lattice L, let ρ_L be the norm defined by

$$\rho_L(x) = \begin{cases} \min\{|\pi|^n : \pi^{-n}x \in L\}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It is clear that the dilation class of ρ_L corresponds to $[L]$ in the realization of the building.

2.2. Independence of vectors.

Definition 2.6: Let $V_1, \ldots, V_n$ be linear subspaces of V, and ρ a norm. We say that the sum $\sum V_i$ is a direct sum modulo ρ if for any $v_i \in V_i$,

$$\rho(v_1 + \cdots + v_n) = sup(\rho(v_1), \ldots, \rho(v_n)). \tag{2.1}$$

This implies in particular that the sum is a direct sum in the usual sense. Note also that this notion depends only on the equivalence class $[\rho]$.

Definition 2.7: The vectors $v_1, \ldots, v_k \in \mathbb{P}(V)$ are called independent modulo ρ if $Kv_1 + \cdots + Kv_k$ is a direct sum modulo ρ.

Definition 2.8: Let $v_1, \ldots, v_{d+1}$ be a basis of V. The apartment $Ap(v_1, \ldots, v_{d+1})$ is the subcomplex of all the $[\rho]$'s such that $v_1, \ldots, v_{d+1}$ are independent modulo ρ.

Since any such ρ is determined by $\rho(v_1), \ldots, \rho(v_{d+1})$, the apartment is homeomorphic to

$$\mathbb{R}^{d+1}/(1, 1, \ldots, 1) \simeq \mathbb{R}^d.$$

Definition 2.9: Let $\sigma = \{[L_0], \ldots, [L_k]\} \in \mathcal{T}$ where $L_0 \supset \cdots \supset L_{k-1} \supset L_k = \pi L_0$. A set of vectors $S \subseteq \mathbb{P}(V)$ is called independent modulo σ if for an arbitrary lift of S of the form $\tilde{S} \subseteq L_0 \setminus \pi L_0$ and for all i, the images of the elements of $\tilde{S} \cap (L_i \setminus L_{i+1})$ in L_i/L_{i+1} are linearly independent over the residue field $O_K/\pi O_K$.

Lemma 2.10: *If $[\rho] \in |\sigma|^o$, independence modulo σ is equivalent to independence modulo ρ.*

Proof: It is convenient to extend the lattice flag periodically to an infinite bi-sided filtration $\cdots L_i \supset L_{i+1} \supset \cdots$. As explained above we can assign numbers $\cdots x_i > x_{i+1} > \cdots$ so that $\rho(v) = x_i$ when $v \in L_i \setminus L_{i+1}$. If the elements of S are independent modulo ρ, then in particular when $v_1, \ldots, v_m \in S$ and

$v_j \in L_i \setminus L_{i+1}$, for any combination $\sum a_j v_j$, where $a_j \in O_K$ but not all are in πO_K, we have $\rho(\sum a_j v_j) = x_i$ and hence $\sum a_j v_j \in L_i \setminus L_{i+1}$. On the other hand, if the elements of S are independent modulo σ, then in verifying (2.1), suppose that all the v_j's lie in L_i but not all in L_{i+1}. Let $S_1 = \{j : v_j \in L_i \setminus L_{i+1}\}$. Then by the assumption, $\sum_{j \in S_1} v_j \in L_i \setminus L_{i+1}$, hence $\sum_j v_j = \sum_{j \notin S_1} v_j + \sum_{j \in S_1} v_j \in L_i \setminus L_{i+1}$ and $\rho(\sum v_j) = x_i = sup_j \rho(v_j)$. ∎

Given a direct sum decomposition $V = \bigoplus_{i=1}^n V_i$, a norm ρ can be rectified so that the sum will be direct modulo the norm. This is done in the following way: let $\pi_{V_i}\colon V \to V_i$ denote the projection, and define

$$\tilde{\rho}(v) = sup(\rho(\pi_{V_1}(v)), \ldots, \rho(\pi_{V_n}(v))).$$

It is easy to see that $\tilde{\rho}$ is a norm, and $V = \bigoplus_{i=1}^n V_i$ modulo $\tilde{\rho}$. Since $\sum \pi_{V_i}(v) = v$, we have for all $v \in V$,

$$\tilde{\rho}(v) \geq \rho(v).$$

THEOREM 2.11: *There is a retraction of* $|\mathcal{T}|$ *to* $\{[\rho] \in |\mathcal{T}| : V = \bigoplus_{i=1}^n V_i \bmod \rho\}$, *carrying* ρ *to* $\tilde{\rho}$.

Proof: Define for $0 \leq t \leq 1$,

$$\rho_t(v) = sup(\rho(v), t \cdot \tilde{\rho}(v)).$$

One has $\rho_0 = \rho, \rho_1 = \tilde{\rho}$. We have to check that this function $([0,1] \times |\mathcal{T}| \to |\mathcal{T}|)$ is continuous, with respect to the metric on $|\mathcal{T}|$ defined by

$$d([\rho_1],[\rho_2]) = log\Big(sup_{x,y\neq 0} \frac{\rho_1(x)}{\rho_2(x)} \frac{\rho_2(y)}{\rho_1(y)}\Big).$$

Let us first prove, for any two norms ρ and η, and $t \in [0,1]$, that

$$d([\rho_t],[\eta_t]) \leq d([\rho],[\eta]). \tag{2.2}$$

Let us denote, for norms α and β,

$$\mu(\alpha,\beta) = sup_{x\neq 0} \frac{\alpha(x)}{\beta(x)},$$

so that $\mu(\alpha,\beta)\mu(\beta,\alpha) = e^{d([\alpha],[\beta])}$. We can assume w.l.o.g. that $\mu(\rho,\eta) = 1$. Let $c = \mu(\eta,\rho)$. Then for any $v \in V$, we have $\rho(v) \leq \eta(v) \leq c\rho(v)$ and $\tilde{\rho}(v) \leq \tilde{\eta}(v) \leq c\tilde{\rho}(v)$. Hence $\rho_t(v) \leq \eta_t(v) \leq c\rho_t(v)$, and (2.2) follows. One can as easily prove that for $t,s \in (0,1]$, $d([\rho_t],[\rho_s]) \leq |log(t/s)|$. This proves continuity away from $t = 0$. Finally, since $\rho(v) \leq \tilde{\rho}(v) \leq e^{d(\rho,\tilde{\rho})}\rho(v)$, we have $\rho(v) \leq \rho_t(v) \leq \rho(v) \cdot sup(1, te^{d(\rho,\tilde{\rho})})$ and hence for t small enough (depending on ρ), $\rho_t = \rho$. ∎

COROLLARY 2.12: *The building $|\mathcal{T}|$ can be retracted onto any of its apartments $Ap(v_1, \ldots, v_{d+1})$. This retraction preserves the independence of any subset of the basis. Namely, if a subset of the basis is independent modulo ρ, it will also be independent modulo ρ_t for $0 \leq t \leq 1$.*

COROLLARY 2.13: *The set of norms ρ such that a given set of linearly independent vectors $v_1, \ldots, v_n$ is independent modulo ρ can be contracted to a point.*

Proof: Denote the above set of norms by I. We first retract I to an apartment of a basis containing $v_1, \ldots, v_n$. The whole apartment is contained in I and remains fixed throughout this retraction. Once inside the apartment, we contract geodesically to an arbitrary point of the apartment. ∎

2.3. PROJECTION OF NORMS. Let $W \subset V$ be a linear subspace, and let ρ be a norm on V. The projection $\bar{\rho} = p_W(\rho)$ is a norm on V/W defined by

$$p_W(\rho)(\bar{v}) = \bar{\rho}(\bar{v}) = inf_{w \in W}\rho(v + w).$$

Since the infimum can equally be taken over the compact set $\{w : \rho(w) \leq \rho(v)\}$, it is attained. From here, one verifies easily that $\bar{\rho}$ is a norm. One can also define a projection of a lattice L to V/W by

$$p_W(L) = \bar{L} = \{\bar{v} : v \in L\}.$$

Since the two projection operations commute with homothety, we have well defined notions of the projections $p_W([\rho])$ and $p_W([L])$.

This allows us to define the projection of a lattice flag, $\sigma = \{[L_0], \ldots, [L_{k-1}]\}$, where $L_0 \supset \cdots \supset L_{k-1} \supset \pi L_0$, by

$$p_W(\sigma) = \bar{\sigma} = \{[\bar{L}_0], \ldots, [\bar{L}_{k-1}]\}. \tag{2.3}$$

$\bar{\sigma}$ is a simplex in the building of V/W, which may be of a smaller dimension than that of σ.

Note that for a lattice L of V, we have the following compatibility relation:

$$p_W(\rho_L) = \rho_{p_W(L)}.$$

We conclude this section with a proposition that will be used later.

PROPOSITION 2.14: *Let $v_1, \dots, v_n \in V$ and let ρ be a norm on V. The following conditions are equivalent:*

(1) *$v_1, \dots, v_n$ are independent modulo ρ.*

(2) *For any $1 \le i < j \le n$, $\bar{v}_i$ and $\bar{v}_j$ are independent modulo $p_{<v_1,\dots,v_{i-1}>}(\rho)$ (the bar stands for reduction modulo $< v_1, \dots, v_{i-1} >$).*

Proof:

$1 \Rightarrow 2$: By the assumption, we have, for any $w \in span(v_i, \dots, v_j, \dots, v_n)$,

$$p_{<v_1,\dots,v_{i-1}>}(\rho)(\bar{w}) = \rho(w),$$

and the independence of $\bar{v}_i$ and $\bar{v}_j$ follows.

$2 \Rightarrow 1$: We will proceed by induction on n. For $n = 1,2$ the claim is obvious. For $n \ge 3$, and $2 \le i < j \le n$, $\bar{v}_i$ and $\bar{v}_j$ are independent modulo $p_{<v_1,\dots,v_{i-1}>}(\rho) = p_{<v_2,\dots,v_{i-1}>}p_{<v_1>}(\rho)$. By the induction hypothesis, $\bar{v}_2, \dots, \bar{v}_n$ are independent modulo $p_{<v_1>}(\rho)$. Hence,

$$p_{<v_1>}(\rho)\left(\sum_{i=2}^{n} a_i\bar{v}_i\right) = sup_{i\ge 2}\{p_{<v_1>}(\rho)(a_i\bar{v}_i)\}.$$

Since v_i and v_1 are independent modulo ρ, $p_{<v_1>}(\rho)(a_i\bar{v}_i) = \rho(a_iv_i)$. Hence

$$p_{<v_1>}(\rho)\left(\sum_{i=2}^{n} a_i\bar{v}_i\right) = sup_{i\ge 2}\{\rho(a_iv_i)\}$$

$$\Rightarrow \forall a_1, \rho\left(a_1v_1 + \sum_{i=2}^{n} a_iv_i\right) \ge sup_{i\ge 2}\{\rho(a_iv_i)\}$$

$$\Rightarrow \rho\left(\sum_{i=1}^{n} a_iv_i\right) = sup_i\{\rho(a_iv_i)\}.$$

The last equality follows since if $\rho(a_1v_1) \le sup_{i\ge 2}\{\rho(a_iv_i)\}$, we have $\rho(\sum_{i=1}^n a_iv_i) \le sup_{i\ge 2}\rho(a_iv_i)$, hence $\rho(\sum_{i=1}^n a_iv_i) = sup_{i\ge 2}\rho(a_iv_i) = sup_{i\ge 1}\rho(a_iv_i)$, and if $\rho(a_1v_1) > sup_{i\ge 2}\{\rho(a_iv_i)\}$, the equality is immediate. ∎

3. The local system $\underline{A}(S)$

3.1. PRELIMINARIES ON GRADED ANTI-COMMUTATIVE ALGEBRAS. We fix an arbitrary field F for the rest of this paper. We will let the coefficients of our algebras lie in this field.

Let us denote by $\mathcal{G}(F)$ the category of graded anti-commutative algebras over the base field F, namely, algebras E with a grading $E = \bigoplus_{n\geq 0} E^n$, such that $E_0 = F$, $E^m E^n \subseteq E^{m+n}$, and for $x \in E^n, y \in E^m$,

$$xy = (-1)^{mn} yx.$$

For $E_1, E_2 \in \mathcal{G}(F)$, their **tensor product** $E_1 \otimes E_2$ is defined as follows: As a vector space, it is the usual vector-space tensor product of E_1 and E_2. The grading is defined in the evident way, and the product structure is defined by

$$(x \otimes y)(x' \otimes y') = (-1)^{mn} xx' \otimes yy'$$

for $x' \in E_1^m, y \in E_2^n$. In other words, when multiplying two expressions we put them together, then let the elements of degree 1 percolate to their place using the relation $xy = -yx$ for them.

3.2. Orlik–Solomon algebras. For any vector space U and a set of lines $S \subseteq \mathbb{P}(U)$ we denote their Orlik–Solomon algebra by $A_{OS}(U, S)$. We recall that it is an object in $\mathcal{G}(F)$, defined as $E(U,S)/I(U,S)$ where $E(U,S)$ is the free exterior algebra on the symbols $\{e_v : v \in S\}$, and $I(U,S)$ is the ideal generated by the elements

$$\delta(e_{v_0} \wedge \cdots \wedge e_{v_k}) := \sum_{i=0}^{k} (-1)^i e_{v_0} \wedge \cdots \wedge \widehat{e_{v_i}} \wedge \cdots \wedge e_{v_k}$$

for all the sequences $v_0, \ldots, v_k \in S$ that are linearly dependent.

Note that S is allowed to be the empty set, and U is allowed to be $\{0\}$; in both cases $A_{OS}(U,S)$ is made of constants only (elements of F), in degree 0.

Being a quotient of $E(U,S)$ by a homogeneous ideal, $A_{OS}(U,S)$ belongs to $\mathcal{G}(F)$.

For $u \in S$ and $S' = S \setminus \{u\}$ we have the restriction–deletion exact sequence

$$0 \to A_{OS}(U, S') \to A_{OS}(U, S) \to A_{OS}(U/<u>, \bar{S}') \to 0. \tag{3.1}$$

(See [O-T], p. 76, theorem 3.65). Note that the latter is only an exact sequence of vector spaces, not even of graded vector spaces since the map from $A(U,S)$ to $A(U/<u>, \bar{S}')$ decreases the grading by 1.

Given a filtration of vector spaces $U = U_0 \supseteq \cdots \supseteq U_k \supseteq U_{k+1} = 0$, we have a surjective map of algebras,

$$A_{OS}(U,S) \to \bigotimes_{i=0}^{k} A_{OS}(U_i/U_{i+1}, \overline{S \cap (U_i \setminus U_{i+1})}) \tag{3.2}$$

sending e_v, $v \in U_i \backslash U_{i+1}$ to $1 \otimes \cdots \otimes 1 \otimes e_{\bar{v}} \otimes 1 \otimes \cdots \otimes 1$. This map is a morphism in $\mathcal{G}(F)$.

3.3. Local systems. Let U be a subcomplex of $\mathcal{T}$.

Definition 3.1: A (cohomological) local system $\underline{A}$ (of abelian groups, vector spaces, etc.) on U is the assignment of an object $A(\tau)$ to any $\tau \in U$, together with maps $r^{\tau_1}_{\tau_2}$: $A(\tau_2) \to A(\tau_1)$ whenever $\tau_1 \geq \tau_2$, satisfying the following compatibility relations: $r^{\tau}_{\tau} = id$, and whenever $\tau_1 \geq \tau_2 \geq \tau_3$, we have $r^{\tau_1}_{\tau_2} r^{\tau_2}_{\tau_3} = r^{\tau_1}_{\tau_3}$.

We denote by $\hat{U}_r \subseteq \hat{\mathcal{T}}_r$ the set of oriented r-simplices of U.

To a local system $\underline{A}$ we attach the differential complex $C^*(U, \underline{A})$ in which $C^r(U, \underline{A})$ for $r \geq 0$ is the collection of maps f: $\hat{U}_r \to \coprod_{\tau \in U_r} A(\tau)$ satisfying $f(\tau) \in A(\tau)$ and $f(\zeta(\tau)) = sgn(\zeta) f(\tau)$, where ζ is the cyclic shift operator.

The differential d: $C^r(U, \underline{A}) \to C^{r+1}(U, \underline{A})$ is defined by

$$df(\tau) = \sum_{i=0}^{r+1} (-1)^i r^{\tau}_{\tau_i} f(\tau_i)$$

where $\tau = (v_0, \ldots, v_{r+1})$ and $\tau_i = (v_0, \ldots, \widehat{v_i}, \ldots, v_{r+1})$.

Definition 3.2: The cohomology of the complex $C^*(U, \underline{A})$ is also referred to as the cohomology of the local system $\underline{A}$ and is denoted by $H^*(U, \underline{A})$. The local system $\underline{A}$ is called acyclic if $H^i(U, \underline{A}) = 0$ for all $i > 0$.

Definition 3.3: The support of a local system $\underline{A}$ is the minimal complex $\mathrm{supp}(\underline{A}) \subseteq U$ such that for any simplex $\sigma \notin \mathrm{supp}(\underline{A})$, $\underline{A}(\sigma) = 0$.

Remark 3.4: If all the restriction maps $r^{\tau_1}_{\tau_2}$ are surjective, then $\mathrm{supp}(\underline{A}) = \{\sigma : A(\sigma) \neq 0\}$. This will always be the case in this article.

Lemma 3.5: *Let $\underline{A}$ be a local system on a complex $U \subseteq \mathcal{T}$. Then we have a canonical isomorphism*

$$H^*(U, \underline{A}) \simeq H^*(\mathrm{supp}(\underline{A}), \underline{A}).$$

Proof: Clear, since canonically $C^*(U, \underline{A}) \simeq C^*(\mathrm{supp}(\underline{A}), \underline{A})$. ∎

3.4. The definition of the local system $\underline{A}(S)$. Let $S \subseteq \mathbb{P}(V)$.

Definition 3.6: The local system $\underline{A}$ (S) on $\mathcal{T}$ is defined by

$$A(S, \sigma) = \bigotimes_{i=0}^{r} A_{OS}(L_i/L_{i+1}, r_{L_i/L_{i+1}}(S_i))$$

where $\sigma = \{[L_0], \dots, [L_r]\}$, $L_0 \supseteq \dots \supseteq L_r \supseteq L_{r+1} = \pi L_0$, $S_i = \tilde{S} \cap (L_i \setminus L_{i+1})$ for an arbitrary lift $\tilde{S} \subseteq L_0 \setminus \pi L_0$ of S with respect to the quotient map from $V \setminus \{0\}$ to $\mathbb{P}(V)$, and $r_{L_i/L_{i+1}}$ is the reduction map from $L_i \setminus L_{i+1}$ to $\mathbb{P}(L_i/L_{i+1})$.

The face maps $A(\sigma, S) \to A(\tau, S)$ for $\sigma \leq \tau$ are given by (3.2).

Let us also recall the definition in [dS]:

Definition 3.7: The local system $\underline{A}_{dS}$ (S) on $\mathcal{T}$ is defined by

$$A_{dS}(S, \sigma) = E(V, S)/I(\sigma, S)$$

where $I(\sigma, S) \triangleleft E(V, S)$ is the ideal generated by the elements

$$\delta(e_{v_0} \wedge \ldots \wedge e_{v_k}) := \sum_{i=0}^{k} (-1)^i e_{v_0} \wedge \cdots \wedge \widehat{e_{v_i}} \wedge \cdots \wedge e_{v_k}$$

for all the sequences $v_0, \dots, v_k \in S$ that are dependent modulo σ.

Remark 3.8: The local system A_{dS} is denoted in [dS] by $\underline{\tilde{A}}$. De Shalit's acyclicity conjecture is formulated for another local system, which is denoted there by $\underline{A}$. However, the two statements are equivalent. See loc. cit., p. 133 and p. 147.

THEOREM 3.9: *We have an isomorphism of local systems*

$$\underline{A}(S) \simeq \underline{A}_{dS}(S).$$

Proof: Preserving the notation of the last two definitions, note that the elements of the form $\delta(e_{[v_0]} \wedge \cdots \wedge e_{[v_k]})$ where all the v_i are in the same S_j, and $v_0, \dots, v_k$ project to a dependent set in L_j/L_{j+1}, suffice to generate the ideal $I(\sigma, S)$.

We first define a map

$$\phi\colon A_{dS}(S, \sigma) \to A(S, \sigma)$$

by $\phi(e_{[v]}) = 1 \otimes \cdots \otimes e_{r_{L_i/L_{i+1}}(v)} \otimes \cdots \otimes 1$ where $v \in S_i$, and e_v appears in the ith place in the tensor product. This map is well defined, by the above observation on the generators of $I(\sigma, S)$.

In the other direction, we define

$$\psi\colon A(S, \sigma) \to A_{dS}(S, \sigma)$$

in the following way: we first define maps $\tilde{\psi}_i\colon E(L_i/L_{i+1}, r_{L_i/L_{i+1}}(S_i)) \to A_{dS}(S, \sigma)$ by $\psi(e_{\bar{v}}) = e_v$, where $v \in S_i$ is an arbitrary lift of the given element $\bar{v} \in r_{L_i/L_{i+1}}(S_i)$. The image does not depend on the choice of a lift since if $v = av' (\mathrm{mod}\, L_{i+1})$ then v and v' are dependent modulo σ, hence $e_{[v']} - e_{[v]} = \delta(e_{[v]} \wedge e_{[v']}) = 0$. It is easy to see that this map vanishes on $I(S, \sigma)$, hence we get a well defined map ψ_i: $A_{OS}(L_i/L_{i+1}, r_{L_i/L_{i+1}}(S_i)) \to A_{dS}(S, \sigma)$. Finally, define $\psi = \bigotimes_i \psi_i$. By their definitions, ϕ and ψ are inverse to one another.

Finally, one must check that the face maps $r^{\tau}_{\tau'}$ correspond to each other, but this is readily seen on the generators. ∎

3.5. A RESTRICTION–DELETION EXACT SEQUENCE FOR $\underline{A}(S)$. We wish to carry over the restriction–deletion exact sequence (3.1) to the context of the local system $\underline{A}$ (S). In doing so we have to tackle two obstacles: One is defining a local system which is analogous to the $A(U/ < u >, S')$ part. The second is that of support. Suppose, for example, that $\sigma = \{[L]\}$ is a vertex, $S = \{[v_1], \ldots, [v_n]\}$, where $v_1, \ldots, v_n \in L \setminus \pi L$, and that we want to take away $[v_n]$. If $r_{L/\pi L}(v_n)$ is equal to one of $r_{L/\pi L}(v_1), \ldots, r_{L/\pi L}(v_{n-1})$, then removing $[v_n]$ does not change the local picture of the arrangement. We cannot expect a short exact sequence similar to (3.1) to be applicable to the quotient $A(S, \sigma)/A(S \setminus \{[v_n]\}, \sigma)$ since in this case, this quotient is 0, whereas any algebra of the form $A(-,-)$ has a nonzero dimension.

For the remainder of this section, let W be a linear subspace of V, and $S \subseteq \mathbb{P}(V) \setminus \mathbb{P}(W)$.

Definition 3.10: The local system $\underline{A}_W(S)$ on $\mathcal{T}$ is defined by

$$A_W(S, \sigma) = A(P_W(S), p_W(\sigma))$$

where P_W: $\mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W)$ is the obvious projection. Namely, $\underline{A}_W(S)$ is the pullback of the system $\underline{A}(P_W(S))$ from the building of V/W, via the projection map (2.3).

PROPOSITION 3.11: *Let $v \in S$ and $S' = S \setminus \{v\}$. The local system $\underline{A}_W(S')$ is embedded in $\underline{A}_W(S)$. If there exists a $w \in S'$ such that $P_W(v) = P_W(w)$ then the local systems $\underline{A}_W(S)$ and $\underline{A}_W(S')$ are identical. Otherwise, $S' \subseteq \mathbb{P}(V) \setminus \mathbb{P}(< W, v >)$, and one has*

$$A_W(S,\sigma)/A_W(S',\sigma) = \begin{cases} 0 & \exists w \in S' : P_W(v), P_W(w) \text{ are} \\ & \qquad \text{dependent} \bmod p_W(\sigma), \\ A_{<W,v>}(S', \sigma) & \text{otherwise.} \end{cases}$$

Proof: The first part of the assertion is obvious. So suppose $P_W(S) \neq P_W(S')$. Hence $S' \subseteq \mathbb{P}(V) \setminus \mathbb{P}(< W, v >)$. Let $p_W(\sigma) = \tau = \{[L_0], \ldots, [L_{k-1}]\}$, where

$L_0, \ldots, L_{k-1}$ are lattices in V/W and $L_0 \supset \cdots \supset L_{k-1} \supset L_k = \pi L_0$. Let $\tilde{S}, \tilde{S}'$ and $\tilde{v}$ be lifts of $P_W(S), P_W(S')$ and $P_W(v)$, respectively, to $L_0 \setminus \pi L_0$. Let i be the index for which $P_W(v) \in L_i \setminus L_{i+1}$. We have

$$A_W(S,\sigma) = \bigotimes_{j=0}^{k-1} A_{OS}(L_j/L_{j+1}, r_{L_j/L_{j+1}}(S_j)), \tag{3.3}$$

$$A_W(S',\sigma) = \bigotimes_{j=0}^{k-1} A_{OS}(L_j/L_{j+1}, r_{L_j/L_{j+1}}(S'_j)), \tag{3.4}$$

where $S_j = \tilde{S} \cap (L_j \setminus L_{j+1})$ and $S'_j = \tilde{S}' \cap (L_j \setminus L_{j+1})$. If there exists $w \in S'$ such that $P_W(v)$ and $P_W(w)$ are dependent modulo τ, then $w \in S_i$ and $r_{L_j/L_{j+1}}(S_j) = r_{L_j/L_{j+1}}(S'_j)$ both for $j \neq i$ (in which case $S_j = S'_j$) and for $j = i$. Hence $A_W(S,\sigma) = A_W(S',\sigma)$ and the quotient is 0.

Otherwise, we have $r_{L_i/L_{i+1}}(S'_i) = r_{L_i/L_{i+1}}(S_i) \setminus \{r_{L_i/L_{i+1}}(\tilde{v})\}$, and by (3.1) we have

$$\begin{aligned} &A_{OS}(L_i/L_{i+1}, r_{L_i/L_{i+1}}(S_i))/A_{OS}(L_i/L_{i+1}, r_{L_i/L_{i+1}}(S'_i)) \\ &\quad \cong A_{OS}(L_i/<L_{i+1},\tilde{v}>, r_{L_i/<L_{i+1},\tilde{v}>}(S'_i)). \end{aligned}$$

For $j \neq i$, $A_{OS}(L_j/L_{j+1}, r_{L_j/L_{j+1}}(S_j)) = A_{OS}(L_j/L_{j+1}, r_{L_j/L_{j+1}}(S'_j))$. Hence the quotient $A_W(S,\sigma)/A_W(S',\sigma)$ is equal to the expression of (3.3), with the ith factor replaced by

$$A_{OS}(L_i/<L_{i+1},\tilde{v}>, r_{L_i/<L_{i+1},\tilde{v}>}(\tilde{S}'_i)).$$

Denoting by $\bar{L}_j$ the further reduction of L_j modulo $<v>$, we have

$$L_i/<L_{i+1},\tilde{v}> \cong \bar{L}_i/\bar{L}_{i+1}$$

and

$$L_j/L_{j+1} \cong \bar{L}_j/\bar{L}_{j+1} \quad \forall j \neq i.$$

This proves the desired equality. ∎

4. The acyclicity theorem

We shall prove the following theorem:

THEOREM 4.1: *For a finite subset $S \subseteq \mathbb{P}(V)$, we have*

$$H^i(\mathcal{T}, \underline{A}(S)) = \begin{cases} 0, & i > 0, \\ A_{OS}(V,S), & i = 0, \end{cases}$$

where the map in degree 0 comes from the collection of natural maps,

$$\{A_{OS}(V,S) \to A_{OS}(L/\pi L, r_{L/\pi L}(\tilde{S}))\}_{[L]\in\mathcal{T}_0}$$

where L is a lattice in V and $\tilde{S}$ is a lift of S to $L \setminus \pi L$.

4.1. Contractible subcomplexes of the building. As we have seen in section 3.5, taking quotients of a local system often yields a smaller support than that of the original system. In this section we shall present a class of couples $(U, \underline{A})$, where $U \subseteq \mathcal{T}$ is a subcomplex and $\underline{A}$ is a local system on U, which is stable under certain operations of taking quotients. We shall also see that the subcomplexes we define are contractible. Together, we shall obtain a machinery for proving the acyclicity theorem by induction.

Definition 4.2: Let $v_1, \ldots, v_n$ be linearly independent elements in $\mathbb{P}(V)$ and let $T_1, \ldots, T_n$ be finite subsets of $\mathbb{P}(V)$ such that $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n$ and $T_i \subseteq \mathbb{P}(V) \setminus \mathbb{P}(< v_1, \ldots, v_i >)$ for all i.

Define a subcomplex $\mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n} \subseteq \mathcal{T}$ by the following condition:

$$\sigma \in \mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n} \Leftrightarrow v_1,\ldots,v_n \text{ are independent } \bmod \sigma, \text{ and}$$
$$\forall 1 \le i \le n, \forall w \in T_i, v_1,\ldots,v_i,w \text{ are independent } \bmod \sigma.$$

Lemma 4.3: *Let $U = \mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n}$, $S \subseteq T_n$ and $v_{n+1} \in S$ (or $n=0$ and $S \subset \mathbb{P}(V)$). Define $S' = S \setminus \{v_{n+1}\}$. Consider the local systems $\underline{A}_{<v_1,\ldots,v_n>}(S)$ and $\underline{A}_{<v_1,\ldots,v_n>}(S')$ on U.*

We have

$$supp_U(\underline{A}_{<v_1,\ldots,v_n>}(S)/\underline{A}_{<v_1,\ldots,v_n>}(S')) = \mathcal{T}^{v_1,\ldots,v_{n+1}}_{T_1,\ldots,T_n,S'}$$

and, denoting the above support by U_{supp},

$$\underline{A}_{<v_1,\ldots,v_n>}(S)/\underline{A}_{<v_1,\ldots,v_n>}(S')|_{U_{supp}} = A_{<v_1,\ldots,v_{n+1}>}(S').$$

Proof: Let $W =< v_1, \ldots, v_n >$. By Proposition 3.11, U_{supp} consists of all the $\sigma \in \mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n}$ such that for any $w \in S'$, $P_W(v_{n+1})$ and $P_W(w)$ are independent modulo $p_W(\sigma)$. For any such w, and $\sigma \in \mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n}$, we know that $v_1, \ldots, v_n, w$ are independent modulo σ (since if $n > 0$, $w \in S \subseteq T_n$). Applying Proposition 2.14, $\sigma \in U_{supp}$ is equivalent (assuming $\sigma \in \mathcal{T}^{v_1,\ldots,v_n}_{T_1,\ldots,T_n}$) to $v_1, \ldots, v_n, v_{n+1}, w$ being independent modulo σ for all $w \in S'$ (note that Proposition 2.14 is applied twice: in its first direction for $v_1, \ldots, v_n, w$ and in the second direction for $v_1, \ldots, v_n, w, v_{n+1}$). Hence $U_{supp} = \mathcal{T}^{v_1,\ldots,v_{n+1}}_{T_1,\ldots,T_n,S'}$.

Moreover, again by Proposition 3.11, the above quotient, restricted to its support, is equal to $\underline{A}_{<v_1,\dots,v_n,v_{n+1}>}(S')$. ∎

THEOREM 4.4: *For $T = \{v_1,\dots,v_n\}, T_1,\dots,T_n$, satisfying the assumptions of Definition 4.2, the complex $\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}$ is contractible. In particular, it is non-empty.*

Proof: The proof will consist of two parts: We will first show that the complex can be continuously retracted into its intersection with a certain apartment, and then we will show that this intersection is contractible.

Let us extend arbitrarily the sequence $v_1,\dots,v_n$ to a basis $v_1,\dots,v_{d+1}$ of V. Recall that we have a projection from $\mathcal{T}$ to the apartment $Ap(v_1,\dots,v_{d+1})$ defined by

$$\tilde{\rho}(x) = sup(\rho(\pi_{v_1}(x)),\dots,\rho(\pi_{v_{d+1}}(x))) \quad \forall[\rho] \in \mathcal{T}$$

and, for any $[\rho] \in \mathcal{T}$, a path from $[\rho]$ to $[\tilde{\rho}]$ defined by

$$\rho_t(x) = sup(\rho(x), t\tilde{\rho}(x)), \quad 0 \le t \le 1.$$

This gives a retracting homotopy, θ: $[0,1] \times \mathcal{T} \to \mathcal{T}$ defined by $\theta(t,[\rho]) = [\rho_t]$. Let us see that this retraction preserves $\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}$.

LEMMA 4.5:

(1) *Let $|\ |$ be a norm in $Ap(v_1,\dots,v_{d+1})$, $1 \le k \le d$ and $w = \sum_{i=1}^{d+1} a_i v_i \in V$. Then $v_1,\dots,v_k,w$ are independent modulo $|\ |$ if and only if*

$$sup_{i\le k}|a_i v_i| \le sup_{i>k}|a_i v_i|. \tag{4.1}$$

(2) *If $w \in V$ and $v_1,\dots,v_k,w$ are independent modulo ρ, then they are also independent modulo $\tilde{\rho}$.*

(3) *For any vectors $u_1,\dots,u_k$ in V, if they are independent both modulo ρ and modulo $\tilde{\rho}$ then they are also independent modulo ρ_t for $0 \le t \le 1$.*

Proof: We adopt the signs $\vee$ and $\wedge$ for max and min, respectively.

(1) We have to check whether for any $b_1,\dots,b_k \in K$,

$$\left|w + \sum_{i=1}^{k} b_i v_i\right| = |w| \vee \bigvee_{i=1}^{k} |b_i v_i|,$$

that is,

$$\left|\sum_{i=1}^{k}(a_i + b_i)v_i + \sum_{i=k+1}^{d+1} a_i v_i\right| = \left|\sum_{i=1}^{d+1} a_i v_i\right| \vee \bigvee_{i=1}^{k} |b_i v_i|.$$

Since $v_1, \ldots, v_{d+1}$ are independent modulo $|\ |$, this is equivalent to

$$(4.2) \qquad \bigvee_{i \leq k} |(a_i + b_i)v_i| \vee \bigvee_{i>k} |a_i v_i| = \bigvee_{i=1}^{d+1} |a_i v_i| \vee \bigvee_{i \leq k} |b_i v_i|.$$

If $\bigvee_{i \leq k} |a_i v_i| \leq \bigvee_{i>k} |a_i v_i|$, then for every $i \leq k$, either $|b_i| \neq |a_i|$ and then $|(a_i + b_i)v_i| = |a_i v_i| \vee |b_i v_i|$, or $|b_i| = |a_i|$, and then $|a_i v_i| \vee |b_i v_i| \leq \bigvee_{i>k} |a_i v_i|$. Thus (4.2) holds. On the other hand, if (4.2) holds then we can choose $b_i = -a_i$, and get (4.1).

(2) Since $\tilde{\rho} \in Ap(v_1, \ldots, v_{d+1})$, we can use the criterion of part 1 of the lemma. Let us denote $w = \sum a_i v_i$. Since $\tilde{\rho}(v_i) = \rho(v_i)$, we have to verify (4.1) for $|\ | = \rho$. Using the assumption, we have

$$\bigvee_{i>k} \rho(a_i v_i) \geq \rho\Big(\sum_{i>k} a_i v_i\Big) = \rho\Big(w - \sum_{i \leq k} a_i v_i\Big)$$
$$= \rho(w) \vee \bigvee_{i \leq k} \rho(a_i v_i) \geq \bigvee_{i \leq k} \rho(a_i v_i).$$

(3) This follows from the definitions and the distributivity of $\vee$ over itself. ∎

Corollary 4.6: *The retraction of $\mathcal{T}$ to $Ap(v_1, \ldots, v_{d+1})$ retracts $\mathcal{T}_{T_1, \ldots, T_n}^{v_1, \ldots, v_n}$ to $\mathcal{T}_{T_1, \ldots, T_n}^{v_1, \ldots, v_n} \cap Ap(v_1, \ldots, v_{d+1})$.*

Finally, we want to contract $\mathcal{T}_{T_1, \ldots, T_n}^{v_1, \ldots, v_n} \cap Ap(v_1, \ldots, v_{d+1})$ to a point. We will actually prove that this set is star-shaped, that is, there is a point in it that "sees" all the other points, and one can contract geodesically to that point.

By part 1 of Lemma 4.5, our set is defined inside $Ap(v_1, \ldots, v_{d+1})$ by finitely many conditions of the form

$$(4.3) \qquad \bigvee_{i \leq k} \rho(a_i v_i) \leq \bigvee_{i>k} \rho(a_i v_i).$$

(Recall that an element ρ of $Ap(v_1, \ldots, v_{d+1})$ is determined by its values on the v_i's.)

This set of conditions can be replaced by a set of inequalities of the form

$$(4.4) \qquad \rho(a_j v_j) \leq \bigvee_{i>k} \rho(a_i v_i)$$

where $j \leq k$. This set of inequalities is bigger than the set we had before, but is still finite. Let us look at a set of stronger inequalities, where each such condition is replaced by a corresponding condition of the form

$$\rho(a_j v_j) \le \bigwedge_{i>k, a_i \neq 0} \rho(a_i v_i).$$

These inequalities define a subset $M \subseteq \mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n} \cap Ap(v_1,\dots,v_{d+1})$. Every point of M sees every point of $\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n} \cap Ap(v_1,\dots,v_{d+1})$. Indeed, if $[\tau] \in M$ and $[\sigma] \in \mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n} \cap Ap(v_1,\dots,v_{d+1})$, for any condition of the form (4.4), there exists an index $i_0 > k$ so that $\sigma(a_j v_j) \le \sigma(a_{i_0} v_{i_0})$, and we also have $\tau(a_j v_j) \le \tau(a_{i_0} v_{i_0})$, since $[\sigma]$ and $[\tau]$ satisfy the same linear inequality, so will any point in the segment connecting them. But this linear inequality implies the inequality (4.4).

It remains to prove that M is nonempty. But a point $[\rho] \in M$ can be chosen by setting $\rho(v_1)$ arbitrarily, then setting $\rho(v_2)$ big enough according to the finitely many inequalities, then $\rho(v_3)$ and so on up to $\rho(v_d)$. ∎

4.2. Proof of the acyclicity theorem. We state a stronger theorem, which is more suitable for an inductive proof.

Theorem 4.7: *Let $n \ge 0$ be an integer. Let $v_1,\dots,v_n$ be linearly independent elements in $\mathbb{P}(V)$ and let $T_1,\dots,T_n$ be finite subsets of $\mathbb{P}(V)$ such that $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n$ and $T_i \subseteq \mathbb{P}(V) \setminus \mathbb{P}(< v_1,\dots,v_i >)$ for all i. Let us write $W =< v_1,\dots,v_n >$.*

Let $S \subseteq T_n$, or $n = 0$ and S is any finite subset of $\mathbb{P}(V)$. Then we have

$$H^k(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) = \begin{cases} 0, & k > 0, \\ A_{OS}(V/W, P_W(S)), & k = 0, \end{cases}$$

where the map in degree 0 comes from the natural maps

$$A_{OS}(V/W, S) \to A_{OS}(p_W(L)/\pi p_W(L), r_{p_W(L)/\pi p_W(L)}(\widetilde{P_W(S)})).$$

This theorem implies Theorem 4.1, via the special case $n = 0$.

Proof: We shall use induction, on the size of S. When S is empty, $\underline{A}_W(S)$ is a constant local system, which assigns the base field F to any simplex. Since the complex $\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}$ is contractible (Theorem 4.4), the cohomology $H^*(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S))$ is 0-dimensional at positive degrees, and one-dimensional at degree 0, as required.

In the inductive step, we can assume that S is nonempty. If two elements of S are equal modulo W, then we can remove one of them from S without changing either the local system $\underline{A}_W(S)$ or the algebra $A_{OS}(V/W, P_W(S))$. Hence, we can assume that all the elements of S are distinct from one another modulo W. Let us choose arbitrarily $v_{n+1} \in S$. Denote $S' = S \setminus \{v_{n+1}\}, W' = < W, v_{n+1} >$. By virtue of Lemma 4.3 we have a short exact sequence

$$0 \to \underline{A}_W(S') \to \underline{A}_W(S) \to \underline{A'}_{W'}(S') \to 0$$

of local systems on $\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}$, where $\underline{A'}_{W'}(S')$ is the local system which has $\mathcal{T}^{v_1,\dots,v_{n+1}}_{T_1,\dots,T_n,S'}$ as its support, and is equal to $\underline{A}_{W'}(S')$ on it. The data

$$((v_1,\dots,v_{n+1}),T_1,\dots,T_n,S')$$

satisfies the assumptions of the theorem (for any $v \in S'$, we have $v \not\equiv v_{n+1} \bmod W$ and hence $v \in \mathbb{P}(V) \setminus \mathbb{P}(W')$).

Let us first prove that $H^i(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) = 0$ for $i > 0$. By the induction hypothesis we have

$$H^i(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S')) = 0$$

and (using Lemma 3.5),

$$H^i(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}'_W(S')) = H^i(\mathcal{T}^{v_1,\dots,v_n,v_{n+1}}_{T_1,\dots,T_n,S'}, \underline{A}_{W'}(S')) = 0.$$

Hence, by the five lemma,

$$H^i(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) = 0.$$

Finally, by the same considerations we have

$$H^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S')) = A_{OS}(V/W, P_W(S'))$$

and

$$H^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}'_W(S')) = H^0(\mathcal{T}^{v_1,\dots,v_n,v_{n+1}}_{T_1,\dots,T_n,S'}, \underline{A}_{W'}(S')) = A_{OS}(V/W', P_{W'}(S')).$$

We get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{OS}(V/W, P_W(S')) & \longrightarrow & A_{OS}(V/W, P_W(S)) & \longrightarrow & A_{OS}(V/W', P_{W'}(S')) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S')) & \to & C^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) & \to & C^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}'_W(S')) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^1(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S')) & \to & C^1(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) & \to & C^1(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}'_W(S')) & \to & 0.
\end{array}$$

In this diagram, all the rows and columns, except the middle column, are exact (the upper row is exact by (3.1)). Hence the middle column is also exact, and $H^0(\mathcal{T}^{v_1,\dots,v_n}_{T_1,\dots,T_n}, \underline{A}_W(S)) = A_{OS}(V/W, P_W(S))$. ∎

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